D' ALEMBERT'S PRINCIPLE

State and explain D' Alembert's principle and hence derive Lagrange's equation for a conservative system.

Ans D' Alembert's principle :- From principle of virtual work we know

$$dW = \sum_i \vec{F_i} \cdot \delta\vec{r_i} = 0$$

for static equilibrium Bernouli extended this principle to a dynamical system. Later on it was advanced by D' Alembert.

Let $\vec{F_i}$ = Force applied on i^{th} particle on the system.

$\vec{P_i}$ = Momentum of the system.

Then from Newton's second law of motion.

$$\vec{F_i} = \frac{d\vec{P_i}}{dt} = \dot{\vec{P_i}}$$

$$\therefore \vec{F_i} - \dot{\vec{P_i}} = 0$$

$\therefore$ Total force on the system $= \Sigma \{\vec{F_i} + (-\dot{\vec{P_i}})\}$

$\sum_i \dot{\vec{P_i}}$ is called kinetic reaction an verse effective force. Then the system is in equilibrium applying principle of virtual work. We have

$$\sum_i (\vec{F_i} - \dot{\vec{P_i}}) \cdot \delta\vec{r_i} = 0$$

This is known as D' Alembert's principle.

From transformation equation we know that

$$\vec{r_i} = \vec{r_i}(q_1, q_2, q_3 - - - - q_n, t) = r_i(q_j$$

$J = 1, 2, 3 - - - -$

$$\delta\vec{r_i} = \sum_J \frac{\partial \vec{r_i}}{\partial q_j} \cdot \delta q_j$$ (since virtual displacement are independent of time)

Then

$$\sum_i (\vec{F_i} - \vec{\dot{p}_i}) \sum_J \left(\frac{\partial \vec{r_i}}{\partial q_J} \cdot \delta q_J\right) = 0$$

or, $$\sum_i \sum_J (\vec{F_i} - \vec{\dot{p}_i}) \cdot \frac{\partial \vec{r_i}}{\partial q_J} \delta q_J = 0$$

or, $$\sum_{i,J} (\vec{F_i} - \vec{\dot{p}_i}) \cdot \frac{\partial \vec{r_i}}{\partial q_J} \cdot \delta q_J = 0$$

or, $$\sum_{i,J} \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_J} \cdot \delta q_J - \sum_{i,J} \vec{\dot{p}_i} \cdot \frac{\partial \vec{r_i}}{\partial q_J} \cdot \delta q_J = 0$$

or, $$\boxed{\sum_{i,J} \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_J} \cdot \delta q_J = \sum_{i,J} \vec{\dot{p}_i} \cdot \frac{\partial \vec{r_i}}{\partial q_J} \cdot \delta q_J}$$

Where $q_J = \vec{F_i} \frac{\partial \vec{r_i}}{\partial q_J} = J^{th}$ component of Generalised force. This is known as D' Alembert's principle. In terms of generalised coordinate.

Lagrange's equation of motion →

Let m_i = Mass of i^{th} particle.

$\vec{r_i}$ = position vector of i^{th} particle

$\therefore m_i \cdot \vec{\dot{r}_i} = \vec{p_i}$

Then from D' Alembert's principle

$$\sum_{i,J} \vec{\dot{p}_i} \frac{\partial \vec{r_i}}{\partial q_J} \cdot \delta q_J = \sum_J q_J \cdot \delta q_J$$

$$\left(\text{where } q_J = \sum_J \vec{F_i} \frac{\partial \vec{r_i}}{\partial q_J}\right)$$

or, $$\sum_{iJ} m_i \vec{\ddot{r}_i} \frac{\partial \vec{r_i}}{\partial q_J} \delta q_J = \sum_J q_J \delta q_J \quad \text{—(1)}$$

$$\therefore \frac{d}{dt}\left(m_i \vec{\dot{r}_i} \frac{\partial \vec{r_i}}{\partial q_J}\right) = m_i \vec{\ddot{r}_i} \frac{\partial \vec{r_i}}{\partial q_J} + m_i \vec{\dot{r}_i} \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_J}\right)$$

$$\text{or, } m_i \vec{\ddot{r}_i} \frac{\partial \vec{r_i}}{\partial q_J} = \frac{d}{dt}\left(m_i \vec{\dot{r}_i} \frac{\partial \vec{r_i}}{\partial q_J}\right) - m_i \vec{\dot{r}_i} \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_J}\right)$$

$$= \sum_{i,J} \frac{d}{dt}\left(m_i \dot{r}_i \frac{\partial \vec{r_i}}{\partial q_J}\right) - \sum_{i,J} m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_J}\right)$$

$$= \sum_{i,J} \frac{d}{dt}\left(m_i \vec{\dot{r}_i} \frac{\partial \vec{r_i}}{\partial q_J}\right) - \sum_{i,J} m_i \vec{\dot{r}_i} \frac{\partial}{\partial q_J}\left(\frac{d \vec{r_i}}{dt}\right)$$

$$\therefore \sum_{i,J} m_i \vec{\ddot{r}_i} \frac{\partial \vec{r_i}}{\partial q_J} = \sum_{i,J} \frac{d}{dt}\left(m_i \dot{r}_i \frac{\partial \vec{r_i}}{\partial q_J}\right) - \sum_{i,J} m_i \vec{\dot{r}_i} \frac{\partial \vec{\dot{r}_i}}{\partial q_J} \quad \text{(11)}$$

$$\because \vec{r_i} = \vec{r_i}(q_J, t)$$

$$\frac{d \vec{r_i}}{dt} = \sum \frac{\partial \vec{r_i}}{\partial q_J} \cdot \frac{d q_J}{dt}$$

$$\vec{\dot{r}_i} = \sum_J \frac{\partial \vec{r_i}}{\partial q_J} \dot{q}_J$$

$$\therefore \sum \frac{\partial \vec{\dot{r}_i}}{\partial \dot{q}_J} = \sum_J \frac{\partial \vec{r_i}}{\partial q_J}$$

$$\frac{\partial \vec{\dot{r}_i}}{\partial \dot{q}_J} = \frac{\partial \vec{r_i}}{\partial q_J}$$

Then from equation (11) ~~of~~ we have

$$\sum_{i,J} m_i \vec{\ddot{r}_i} \frac{\partial \vec{r_i}}{\partial q_J} = \sum_{i,J} \frac{d}{dt}\left(m_i \vec{\dot{r}_i} \frac{\partial \vec{\dot{r}_i}}{\partial \dot{q}_J}\right) - \sum_{i,J} m_i \vec{\dot{r}_i} \frac{\partial \vec{\dot{r}_i}}{\partial q_J}$$

$$= \sum_J \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_J}\left(\sum \frac{1}{2} m_i \dot{r}_i^2\right) - \sum_J \frac{\partial}{\partial q_J}\left(\frac{1}{2} m_i \dot{r}_i^2\right)\right)$$

Or, $\sum_{i,J} m_i \dot{\vec{r}_i} \frac{\partial \vec{r_i}}{\partial q_J} = \sum_J \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_J} - \sum_J \frac{\partial T}{\partial q_J}$

Combaning equation (i) and (iii)

$$\left\{ \sum_J \frac{d}{dt} \cdot \frac{\partial T}{\partial \dot{q}_J} - \sum_J \frac{\partial T}{\partial q_J} \right\} \delta q_J = \sum_J Q_J \cdot \delta q_J.$$

$$\cancel{\sum_J} \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_J} - \frac{\partial T}{\partial q_J} \right\} \cdot \cancel{\delta q_J} = \cancel{\sum_J} Q_J \cdot \cancel{\delta q_J}$$

$$\frac{d}{dt} \cdot \frac{\partial T}{\partial \dot{q}_J} - \frac{\partial T}{\partial q_J} = Q_J$$

If V = potential energy of system

$$\therefore Q_J = -\frac{\partial V}{\partial q_J}$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_J} - \frac{\partial T}{\partial q_J} = -\frac{\partial V}{\partial q_J}$$

or, $\frac{d}{dt} \cdot \frac{\partial T}{\partial \dot{q}_J} - \frac{\partial (T-V)}{\partial q_J} = 0$

put $L = T - V$ $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_J} - \frac{\partial L}{\partial q_J} = 0$

$$\therefore \frac{\partial L}{\partial \dot{q}_J} = \frac{\partial T}{\partial \dot{q}_J}$$

Since P.E. is independet of velocity.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_J} - \frac{\partial L}{\partial q_J} = 0 \qquad \text{(iv)}$$

This is Lagrange equation of motion of holonomic Conservation system.

$L = T - V$ = Lagrange of the System

$$L = L(q_J, \dot{q}_J, t).$$

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